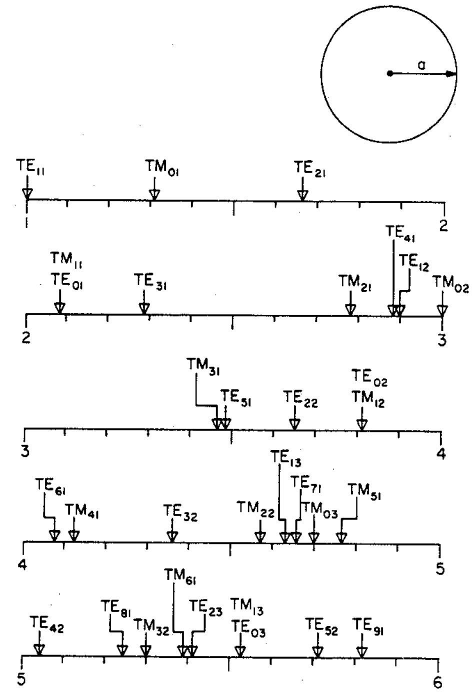

Fig. 5. Normalized modal cutoff frequencies for a circular waveguide.

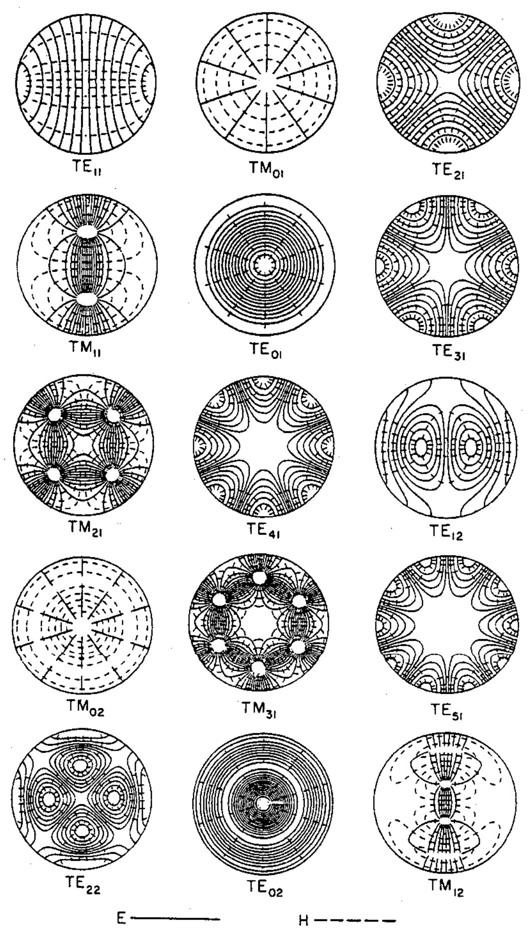

Fig. 6. Transverse modal field distribution for a circular waveguide (first 30 modes).

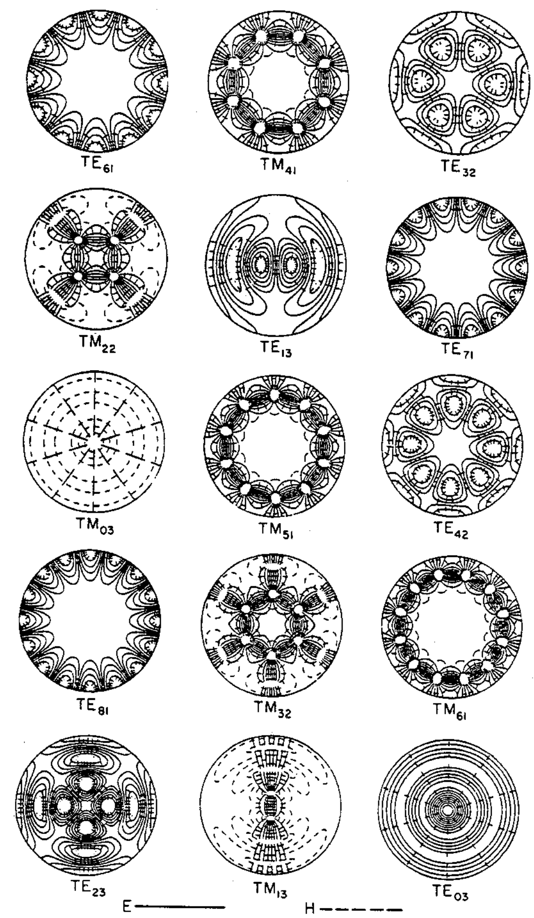

Fig. 6. (*Continued*)

The density of the field lines is approximately proportional to the field strength. These plots are done by a Cyber 175 computer.

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Computations of Frequencies and Intrinsic Q Factors of TE_{0nm} Modes of Dielectric Resonators

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***Abstract*—The Rayleigh–Ritz method is described, which is used to calculate the resonant frequencies and intrinsic Q factors due to dielectric**

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losses of quasi TE_{0nm} modes of dielectric resonators. Electromagnetic fields of an auxiliary post dielectric resonator are taken as an electrodynamic basis for approximate solutions of this problem.

The method provides upper bounds for true resonant frequencies. Numerical results are compared with previously published complementary calculations. The influence of a dielectric substrate on resonant frequencies and intrinsic Q values is demonstrated.

I. Introduction

Dielectric resonators have found many practical applications due to their advantages of low cost, small size, and good temperature stability. They are particularly used in microwave integrated circuits (MIC's). A typical structure of an MIC dielectric resonator is shown in Fig. 1. This paper describes the analysis of nonradiating quasi-TE_{0nm} modes of this structure. Many methods for the calculations of resonant frequencies of cylindrical dielectric resonators have been presented. Accurate calculations by the separation of variables method are available only for a dielectric post resonator. Analyses have been performed by B. Hakki and P. Coleman [1] for nonradiating modes and by Y. Kobayashi and S. Tanaka [2] for all modes. If the height of the dielectric resonator is smaller than the distance between the metallic plates, only approximate solutions are available. For this case, models such as a magnetic wall model [3], [4], a dielectric waveguide model [5], [6], and a mixed model [7] have been used. However, the most accurate results have been obtained by the Weinstein variational method [8] and the method of matching of modal expansions between two complementary cylindrical regions [9]. Interesting methods of analysis of dielectric resonators in free space have been presented in [10]–[13]. Those methods are useful for radiating modes when the distance between the metallic plates is much greater than the height of the dielectric resonator. In this paper, a method using the Rayleigh–Ritz formalism is presented. The method is accurate in the sense that resonant frequencies and intrinsic Q factors due to dielectric losses converge to their exact values when the number of terms in the expansion increases to infinity.

II. Analysis

Analyses have been performed under the assumption that the distance L between the conducting plates is smaller than half of the wavelength in region II of Fig. 1. This is necessary because we are looking for nonradiating modes only. According to the Rayleigh–Ritz method, the electromagnetic field is expanded into a series of basis functions. For our problem, the electrodynamic basis is formed as a set of functions being the solutions of an auxiliary eigenvalue problem for a dielectric post resonator. The post resonator has the height equal to L, the radius equal to a, and real permittivity ϵ_b. It is evident that only nonradiating TE_{0nm} modes of the post resonator participate in the expansion of fields for the MIC dielectric resonator structure. They are purely rotational modes so the system of linear equations obtained by the Rayleigh–Ritz method takes the following form [14], [15]:

$$\left(A_{ij} - \delta_{ij}\frac{1}{\hat{\omega}^2}\right)\alpha_j^{(H)} = 0, \qquad i,j = 1,2,\cdots,N \tag{1}$$

where

$\{\alpha_j^{(H)}\}$ set of coefficients of the magnetic field expansion to be determined,

N number of basis functions,

$\hat{\omega}$ complex angular frequency to be determined,

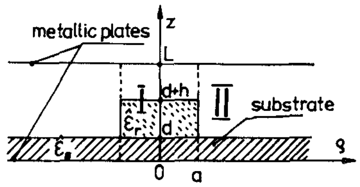


Fig. 1. Structure of the MIC dielectric resonator.

and

$[A]$ matrix with elements given by the following expression:

$$A_{ij} = \frac{\langle \hat{\epsilon}_r(\rho,z)\vec{E}_i, \vec{E}_j\rangle}{\omega_i\omega_j} \tag{2}$$

where

$$\epsilon_r(\rho,z) = \begin{cases} \hat{\epsilon}_s \text{ for } 0 < z < d \\ \hat{\epsilon}_r \text{ for } d < z < d+h \text{ and } 0 < \rho < a \\ 1 \text{ for remaining part of the region} \end{cases}$$

$$\langle \epsilon_r(\rho,z)\vec{E}_i, \vec{E}_j\rangle \underset{df}{=} \int_0^\infty \left[\rho\int_0^L \hat{\epsilon}_r(\rho,z)\vec{E}_i\cdot\vec{E}_j^*\,dz\right]d\rho$$

$\{\vec{E}_i\}$ set of TE_{0nm} electric fields of the post resonator

$\{\omega_i\}$ set of TE_{0nm} angular frequencies of the post resonator.

The electric fields corresponding to the TE_{0nm} modes of the post dielectric resonator can be written as follows:

$$\vec{E}_i = \begin{cases} B_i J_1\left(k_i^{(\epsilon)}\cdot\rho\right)\sin\left(\dfrac{m\pi}{L}z\right)\vec{i}_\varphi & \text{for } \rho < a \\ B_i\dfrac{J_1\left(k_i^{(\epsilon)}\cdot a\right)}{K_1\left(k_i^{(0)}\cdot a\right)}K_1\left(k_i^{(0)}\rho\right)\sin\left(\dfrac{m\pi}{L}z\right)\vec{i}_\varphi & \text{for } \rho \geqslant a. \end{cases} \tag{3}$$

Because the following orthogonality relation holds:

$$\langle \epsilon_b(\rho,z)\vec{E}_i, \vec{E}_j\rangle = \delta_{ij} \tag{4}$$

where

$$\epsilon_b(\rho,z) = \begin{cases} \epsilon_b \text{ for } \rho < a \\ 1 \text{ for } \rho > a \end{cases}$$

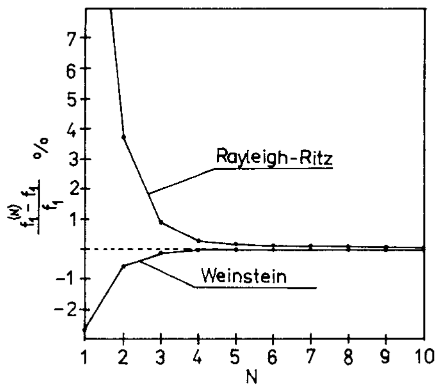
so

$$B_i^2 = 2\Bigg/\left\{L\left[\epsilon_b\int_0^a \rho J_1^2\left(k_i^{(\epsilon)}\cdot\rho\right)d\rho + \left(\frac{J_1\left(k_i^{(\epsilon)}\cdot a\right)}{K_1\left(k_i^{(0)}\cdot a\right)}\right)^2\int_0^\infty \rho K_1^2\left(k_i^{(0)}\cdot\rho\right)d\rho\right]\right\}. \tag{5}$$

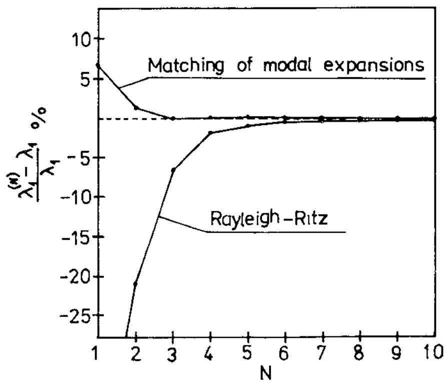
The sets $\{\omega_i\}$, $\{k_i^{(\epsilon)}\}$, $\{k_i^{(0)}\}$ are found as roots of the following system of equations:

$$\begin{cases} \dfrac{ak_i^{(\epsilon)}J_0\left(k_i^{(\epsilon)}\cdot a\right)}{J_1\left(k_i^{(\epsilon)}\cdot a\right)} + \dfrac{ak_i^{(0)}K_0\left(k_i^{(0)}\cdot a\right)}{K_1\left(k_i^{(0)}\cdot a\right)} = 0 \\ k_i^{(\epsilon)} = \left[\left(\dfrac{\omega_i}{c}\right)^2\epsilon_b - \left(\dfrac{m\pi}{L}\right)^2\right]^{1/2} \\ k_i^{(0)} = \left[\left(\dfrac{m\pi}{L}\right)^2 - \left(\dfrac{\omega i}{c}\right)^2\right]^{1/2} \end{cases} \tag{6}$$

where $J_n(x)$ denotes the Bessel functions of the first kind, $K_n(x)$ is the modified Hankel functions of the second kind, and c the light velocity.



(a)



(b)

Fig. 2. Convergence of approximate values: (a) quasi-TE_{011}-mode frequency, (b) quasi-TE_{011}-mode wavelength of dielectric resonators having (a) $h = 2.14$ mm, $a = 3.995$ mm, $L = 11.0$ mm, $\epsilon_r = 36.2$, $d = 4.43$ mm, $\epsilon_s = 1$ and, $\epsilon_b = \epsilon_r$. (b) $h = 13.37$ mm, $a = 8.995$ mm, L-46.40 mm, $\epsilon_r = 34\,61$, $d = 0$, $\epsilon_s = 1$, and $\epsilon_b = \epsilon_r$.

TABLE I

THE COMPUTED VALUES OF FREQUENCY OF THE QUASI-TE_{011} MODE VERSUS NUMBER N OF BASIS FUNCTIONS FOR THE RESONATOR HAVING $h = 2.14$ mm, $a = 3.995$ mm, $L = 11\,0$ mm, $\epsilon_r = 36.2$, $d = 4.43$ mm, $\epsilon_s = 1$, AND $\epsilon_b = \epsilon_r$

	Subscripts of basis		$f_1^{(N)}$ MHz	$f_1^{(N)}$ MHz ref. [5]
N	n	m	Rayleigh-Ritz	Weinstein
1	1	1	9022.4	7551.9
2	1	3	8056.9	7719.5
3	1	5	7832.8	7752.4
4	1	7	7783.5	7757.8
5	1	9	7777.3	7758.3
6	2	3	7774.0	7758.3
7	2	5	7770.8	7758.4
8	2	1	7768.8	7758.5
9	1	15	7766.9	7758.5
10	1	13	7765.5	7758.5
20			7762.3	—
Measured ref. [5]				7790.0
Itoh model ref. [5]				8380.0
Fiedziuszko model ref. [5]				7160.0

TABLE II

THE COMPUTED VALUES OF WAVELENGTH OF THE QUASI-TE_{011} MODE VERSUS NUMBER N OF BASIS FUNCTIONS FOR THE RESONATOR HAVING $h = 13.37$ mm, $a = 8.995$ mm, $L = 46.40$ mm, $\epsilon_r = 34.61$, $d = 0$, $\epsilon_s = 1$, AND $\epsilon_b = \epsilon_r$

	Subscripts of basis		$\lambda_1^{(N)}$ m	$\lambda_1^{(N)}$ m
N	n	m	Rayleigh-Ritz	ref. [6]
1	1	1	0.057364	0.110941
2	1	2	0.082170	0.105237
3	1	3	0.097165	0.103873
4	1	4	0.102134	0.103978
5	1	5	0.102897	0.104035
6	2	2	0.103351	0.103987
7	1	7	0.103533	0.103947
8	1	8	0.103644	—
9	1	11	0.103680	0.103943
10	1	10	0.103707	—
17			0.103850	0.103929

Substituting (3) in (2), one obtains

$$A_{ij} = \Bigg\{ \delta_{ij} + B_i B_j \Bigg[\int_0^a \rho J_1\big(k_i^{(\epsilon)} \cdot \rho\big) J_1\big(k_j^{(\epsilon)} \cdot \rho\big)\, d\rho \bigg((\hat{\epsilon}_r - \epsilon_b) \int_d^{d+h} \sin\Big(\frac{m\pi}{L} z\Big) \sin\Big(\frac{m'\pi}{L} z\Big)\, dz$$
$$+ (1 - \epsilon_b) \int_{d+h}^{L} \sin\Big(\frac{m\pi}{L} z\Big) \sin\Big(\frac{m'\pi}{L} z\Big)\, dz + (\hat{\epsilon}_s - \epsilon_b) \int_0^d \sin\Big(\frac{m\pi}{L} z\Big) \sin\Big(\frac{m'\pi}{L} z\Big)\, dz \bigg)$$
$$+ \frac{J_1\big(k_i^{(\epsilon)} \cdot a\big) J_1\big(k_j^{(\epsilon)} \cdot a\big)}{K_1\big(k_i^{(0)} \cdot a\big) K_1\big(k_j^{(0)} \cdot a\big)} \int_a^\infty \rho K_1\big(k_i^{(0)} \cdot \rho\big) K_1\big(k_j^{(0)} \cdot \rho\big)\, d\rho \cdot (\hat{\epsilon}_s - 1) \int_0^d \sin\Big(\frac{m\pi}{L} z\Big) \sin\Big(\frac{m'\pi}{L} z\Big)\, dz \Bigg] \Bigg\} \Big/ \omega_i \Big/ \omega_j. \tag{7}$$

Solving the system of equations (1), one obtains the set of eigenvalues $\{\mu_i\}\, i = 1, 2, \cdots, N$ and N corresponding eigenvectors being the magnetic-field expansion coefficients for different quasi-TE_{0nm} modes. Then the resonant frequencies f_i and the intrinsic Q_i values due to dielectric losses can be found as follows:

$$\hat{\omega}_i = 1/\sqrt{\hat{\mu}_i} \tag{8}$$

$$fi = \mathrm{Re}(\hat{\omega}_i)/2\pi \tag{9}$$

$$Q_i = \mathrm{Re}(\hat{\omega}_i)/2Im(\hat{\omega}_i). \tag{10}$$

When dielectric losses are not too high, they do not affect the f_i values (this holds for practical cases). Then, according to general features of the Rayleigh–Ritz method [16], the consecutive expansion of the matrix $[A]$ dimension yields decreasing sequences of frequencies. It means that the method provides upper bounds for the true resonant frequencies of the quasi-TE_{0nm} modes. Particularly for the lowest quasi-TE_{011} mode, we have

$$f_1^{(1)} \geqslant f_1^{(2)} \geqslant \cdots \geqslant f_1^{(N)} > f_1 \tag{11}$$

where f_1 is the true resonant frequency and $f_1^{(N)}$ is the approximate value obtained with N basis functions.

TABLE III
THE COMPUTED VALUES OF FREQUENCIES AND INTRINSIC Q FACTORS FOR DIELECTRIC MIC RESONATORS HAVING $\epsilon_r = 36.2(1 - j\,10^{-4})$, $L = 11.0$ mm, AND $\epsilon_b = \mathrm{Re}(\hat{\epsilon}_r)$

	Permittivity	Dimensions			TE_{011} modes		TE_{012} modes	
	$\hat{\epsilon}_s$	h mm	a mm	d mm	$f_1^{(20)}$ MHz	$Q_1^{(20)}$	$f_2^{(20)}$ MHz	$Q_2^{(20)}$
1	$3.03 - j1.65 \times 10^{-2}$	2.14	3.995	1.77	8015.2	5520	13528.0	6130
2	$1 - j0$	2.14	3.995	1.77	8056.5	10320	13582.8	10327
3	$3.03 - j1.65 \times 10^{-2}$	4.16	3.015	1.77	8010.4	9970	11637.2	9616

III. COMPUTATIONS RESULTS

Computations of the quasi-TE_{011}-mode resonant frequency of dielectric resonators having the same dimensions and permittivities as in [8] and [9] have been carried out first. The results are presented in Fig. 2(a) and in Table I for the M. Jaworski *et al.* resonator [8] and in Fig. 2(b) and in Table II for that of D. Maystre *et al.* [9]. Fig. 2 shows the convergence of approximate solutions versus the number of basis functions. The convergence of the Rayleigh–Ritz method is worse than Weinstein's and the matching of modal expansions methods. On the other hand, the Rayleigh–Ritz method leads to a simple eigenvalue problem which can be solved faster than the problem of the vanishing determinant which must be solved when those methods are used. For the Rayleigh–Ritz method, basis functions have been chosen in the way described in [14] to get the fastest convergence of the quasi-TE_{011}-mode frequency. Subscripts of the basis functions are shown in Tables I and II. An important feature of the Rayleigh–Ritz method is that it provides upper bounds for true resonant frequencies. Therefore, it is complementary to the Weinstein method, which provides lower bounds for them.

For the proof of this see, e.g., [16]. It can be seen from Table II and Fig. 2(b) that the method of matching of modal expansions also provides lower bound for the quasi-TE_{011}-mode frequency.

Using two complementary methods, one can easily assess the maximum error of calculations of resonant frequencies. It is smaller than half of the difference between the values obtained by these methods.

As the second example, the values of resonant frequencies and intrinsic Q factors of full MIC dielectric resonators, shown in Fig. 1, have been computed. The results are presented in Table III. The first two lines in this table show the influence of the substrate on frequencies and Q values. It is seen that the substrate changes the frequencies less that by 1 percent (note that the dielectric constant of the substrate is low). The influence of substrate losses on the Q values is considerable. For a lossless substrate, the intrinsic Q values are approximately equal to the reciprocals of $\tan\delta$ of the dielectric resonator medium, while for a lossy substrate, the Q values decrease 40 percent. The influence of substrate losses is greater when the h/a ratio is smaller (compare lines 1 and 3 from Table III).

IV. CONCLUSIONS

Accurate values of resonant frequencies and intrinsic Q factors of MIC dielectric resonators could be found by the Rayleigh–Ritz method using electromagnetic fields of a post dielectric resonator as an electrodynamic basis. The method described in this paper allows one to find nonradiating quasi-TE_{0nm} modes. For low-loss resonators, the method provides upper bounds for true resonant frequencies. Application of the method requires the solution of an eigenvalue problem for a complex matrix of not very high order. Upper and lower bounds of resonant frequencies can be assessed if two complementary methods are used for calculations.

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Cross-Coupled Coaxial-Line/Rectangular-Waveguide Junction

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***Abstract*—The analysis of a cross-coupled coaxial-line/rectangular-waveguide junction having dissimilar coaxial lines is presented. An equivalent circuit is deduced for the case where the TE_{10} mode is the only propagating waveguide mode. Experimental/theoretical comparisons are also reported which show the analysis to be very accurate.**

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